

## Modified pure spinors and mirror symmetry

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### Abstract

It has been argued recently that mirror symmetry exchanges two pure spinors characterizing a generic manifold with  $SU(3)$ -structure. We show how pure spinors are modified in the presence of topological D-branes, so that they are still exchanged by mirror symmetry. This exchange emerges from the fact that the modified pure spinors come out as moment maps for the symmetries of A and B-models. The modification by the gauge field is argued to ensure the inclusion into the mirror exchange of the A-model non-Lagrangian branes endowed with a non-flat connection. Treating the connection as a distribution on an ambient six-manifold, assumed to be  $T^3$ -fibered, the proposed mirror formula is established by fiberwise T-duality.

# 1 Introduction and conclusions

Generalized geometry [1, 2], which naturally incorporates the action of a two-form  $B$  and has played an important role in number of recent string-theoretic applications, can be defined in terms of formal sums of special even or odd forms called pure spinors. For any manifold of  $SU(3)$ -structure there is a pair of such objects, the exponentiated fundamental form  $e^{i\omega}$  and the holomorphic three-form  $\Omega$ , and the action of mirror symmetry in string backgrounds (generally speaking with fluxes) can be described as their exchange. Incorporating D-branes in this picture is of obvious physical interest. In the large-volume geometrical approach these can be viewed as gauge bundles supported over some submanifold  $Y$  of a manifold  $X$ . In the framework of topological strings, D-branes must satisfy some stability condition in order to correspond to extended objects of untwisted string theory. Incidentally, the D-brane stability conditions can be compactly written using the pure spinors.

Just as for the pure spinors, the natural distinguishing principle for D-branes is whether the volume form of the (sub-)manifold, wrapped by the brane, is of odd or even rank. For the even-dimensional, or B-type, branes one finds on the world-volume the deformed Hermitian Yang–Mills equations

$$\mathrm{Im}(e^{i\theta} e^{i\omega+F}) = 0. \quad (1.1)$$

These were obtained in [3] by considering open-string effective actions and studying BPS conditions; similar deformed equations have emerged in differential-geometric studies of stability of vector bundles [4, 5]. They correspond to the vanishing of a moment map derived from certain symplectic structures on the space of gauge connections. On the other hand, world-sheet techniques allow for the identification of topological branes with solutions of current-matching equations [6]. Stability is demanded in this context by matching spectral flows [7], which not only confirms the stability equations obtained through space-time techniques for holomorphic line bundles, but yields stability equations for A-branes, with possible involvement of a non-trivial field strength:

$$\mathrm{Im}(e^{i\theta} \Omega|_Y \wedge F^l) = 0, \quad (1.2)$$

where  $Y$  is the world-volume of the A-brane and  $l$  is related to its transverse complex dimension. The case of non-zero  $l$  corresponds to the coisotropic branes discovered by Kapustin and Orlov [8], whereas  $l = 0$  corresponds to the more familiar Lagrangian branes. If one considers an A-brane  $Y$  inside a Calabi–Yau three-fold  $X$ , it comes with a bundle whose curvature descends to a two-form of the quotient bundle  $\mathcal{F}Y$  defined by the following exact sequence:

$$0 \rightarrow \mathcal{L}Y \hookrightarrow TY \rightarrow \mathcal{F}Y \rightarrow 0, \quad (1.3)$$

where  $\mathcal{L}Y = \ker \omega|_Y$ . The A-type boundary conditions mix the metric and the field strength so that  $\mathcal{L}Y$  is annihilated by the field strength. Moreover,  $\omega^{-1}F$  defines a complex structure on the quotient bundle  $\mathcal{F}Y$ . The complex dimension of  $\mathcal{F}Y$  is precisely twice the integer  $l$  encountered above.

The condition of a Lagrangian cycle  $L$  to be special is a BPS condition: it consists of the existence of a constant phase  $e^{i\theta}$  such that the ambient holomorphic form, once rotated by this phase, pulls back to a volume form on the cycle  $L$ :

$$\Omega|_L = e^{i\theta} \mathrm{vol}_L. \quad (1.4)$$

The phase  $\theta$  encodes the combination of the two supersymmetry generators that gives rise to the unbroken supersymmetry [9]. Lagrangian branes satisfying the stability equation, or special Lagrangian branes, were shown [10] to lead to stable holomorphic line bundles under Fourier–Mukai transform.

The present paper aims at including the general coisotropic branes in this mirror-symmetric picture, thus identifying non-Lagrangian A-branes with non-zero field strength also as mirror counterparts of B-branes. A way of tying these together with the Lagrangian branes is to make them fit into the symplectic framework, that was developed on the A-side by Thomas [11, 12]. In [10, 12] a mirror exchange of moment-map problems for A and B models was proposed. We find a realization of these moment maps as modified pure spinors, which allows for extensions of the mirror picture, by inclusion of not only non-Lagrangian A-branes but also non-vanishing  $B$ -field.

Indeed one would like to show that there exists a modification of pure spinors by terms involving the field strength  $F$ , that yields both the B-type and A-type stability conditions, (1.1) and (1.2) respectively, and that the modified pure spinors are explicitly exchanged under the mirror map. As they correspond to maximally isotropic spaces, the pure spinors are annihilated by half of gamma-matrices of Clifford(6,6) algebra. Their similarity with the Ramond–Ramond fields, which are also spinors of Clifford(6,6) (see [13]) has already been used for the quantitative mirror symmetry proposals in flux backgrounds [14, 15]. In the study of the mirror symmetry with D-branes, we can use the Clifford(6,6) picture of D-brane charges [16]; note that this can be done with a non-zero  $B$ -field (and  $H$ -flux). From other side, when turning the  $H$ -flux off, we should find agreement with previous results concerning mirror symmetry on Calabi–Yau manifolds with bundles [10, 12, 17]. In that sense, turning the flux off provides additional consistency checks for the generalized mirror conjectures [14, 18, 19, 20]. Note that while for the closed-string case the exchange of pure spinors under mirror symmetry was verified under the assumption that the ambient manifold is  $T^3$ -fibered, it was conjectured and verified by turning Ramond–Ramond fluxes on and following their transformations, that this assumption may be eventually dropped. Here we will be verifying the exchange of the modified pure spinors on the  $T^3$ -fibered manifolds.

The structure of the paper is as follows. In section 2 we consider the simple case of D-branes on  $T^6$ . There are no new results here, but this example provides a simple illustration of more general results presented in the bulk of the paper. There we will work formally, considering bundles supported on a submanifold  $Y$  of a Calabi–Yau three-fold  $X$ . In section 3 we extend the treatment of the special Lagrangian A-branes to generic coisotropic ones. In section 4 we present the modification of pure spinors and consider the action of T-duality; we also include some concluding speculations on the underlying closed-open string picture and non-Abelian gauge fields.

## 2 From holomorphic line bundles to non-Lagrangian branes

The ref. [10] establishes the exchange between special Lagrangian branes and stable B-branes under Fourier–Mukai transform, starting from special Lagrangian A-branes. In particular, the input data involve a flat connection, and therefore the curvature on the B-brane comes entirely from the shift in the connection induced by the integrand in the Fourier–Mukai transform. The coisotropic A-branes are simply excluded from this picture. Reversing the process of the derivation and starting from holomorphic line bundles in the B-model, one may hope for the appearance of both Lagrangian and non-Lagrangian branes after performing Fourier–Mukai transform.

In the simplest example of a six-torus, it was observed in [21] how the curvature of a holomorphic line bundle influences the dimensionality of the brane obtained by Fourier–Mukai transform. This computation is instructive and some of its outputs (such as the dimensionality of the mirror brane) can be extended to other Calabi–Yau manifolds, assuming the existence of a  $T^3$ -fibration along the lines of the Strominger–Yau–Zaslow argument [22]. We therefore review it to set the stage for the fiberwise T-duality that we will use in section 4, in the framework of generalized complex geometry (see also [23, 24]). Let us regard  $T^6$  as a trivial fibration  $T^3 \times T^3$ , put coordinates  $x^\mu$  on the first factor,  $y^\mu$  on the second one, which we regard as the fiber. We pick a complex structure by writing

$z^\mu = x^\mu + iy^\mu$ , and consider a holomorphic line bundle on  $T^6$ , or equivalently the curvature

$$F = F_{\mu\nu} dz^\mu d\bar{z}^\nu,$$

viewed as a distribution whose support is the B-brane we start with. On a  $T^3$ -fibered Calabi–Yau manifold, we could follow this path in a local sense, with  $x^\mu$  being a coordinate in a local chart on the base. Going to real coordinates makes the antisymmetric part of the matrix  $(F_{\mu\nu})$  appear in the diagonal blocks, and its symmetric part in the anti-diagonal blocks.

$$\begin{aligned} F &= F_{\mu\nu}(dx^\mu \wedge dx^\nu - dy^\mu \wedge dy^\nu) + iF_{\mu\nu}(dx^\mu \wedge dy^\nu + dy^\mu \wedge dx^\nu) \\ &= F_{\mu\nu}(dx^\mu \wedge dx^\nu - dy^\mu \wedge dy^\nu) + iF_{\mu\nu}(dx^\mu \wedge dy^\nu - dy^\mu \wedge dx^\nu) \\ &= F_{\mu\nu}^{(A)}(dx^\mu \wedge dx^\nu - dy^\mu \wedge dy^\nu) + iF_{\mu\nu}^{(S)}(dx^\mu \wedge dy^\nu). \end{aligned}$$

As antisymmetric matrices, the diagonal blocks have even rank (moreover holomorphicity implies that the two ranks are equal). The case where this rank is zero is precisely the one considered in [10], while the case of rank two leads to a codimension-one A-brane, as we are going to see.

Let us rewrite the curvature as

$$F = \mathcal{A}_{\mu\nu}(dx^\mu \wedge dx^\nu + dy^\mu \wedge dy^\nu) + \mathcal{S}_{\mu\nu}dx^\mu \wedge dy^\nu,$$

and note at this point that a mere counting argument would yield the relationship between this block-structure and the dimension of the T-dual A-brane. If the antisymmetric block  $\mathcal{A}$  was absent, the B-brane would have a four-dimensional world-volume spanning two directions in the basis and two in the fiber. This configuration is T-dual to a three-dimensional world-volume, as we shall see by Fourier transforming the expression. Turning on  $\mathcal{A}$  has the effect of mixing the  $x$  and  $y$  coordinates of the support of the curvature, so that fiberwise T-duality removes one directions and adds two. Going through the Fourier–Mukai transformation yields a distribution on the dual fibration  $T^3 \times \tilde{T}^3$ , with the dual fiber equipped with coordinates  $\tilde{y}$ .

$$\begin{aligned} e^{F'} &= \int_{T_y^3} e^{dy_\mu d\tilde{y}^\mu} e^F \\ &= e^{\mathcal{A}_{\mu\nu}(dx^\mu \wedge dx^\nu)} \int_{T_y^3} e^{dy_\mu d\tilde{y}^\mu} e^{\mathcal{A}_{\mu\nu}dy^\mu \wedge dy^\nu + \mathcal{S}_{\mu\nu}dx^\mu \wedge dy^\nu}. \end{aligned}$$

If  $\mathcal{A}$  is zero, then the curvature  $F'$  is also zero, and the (Poincaré dual of the) resulting Chern character is proportional to the distribution

$$\delta(\tilde{y}_\mu - \mathcal{S}_{\mu\nu}x^\nu),$$

which situation corresponds to a Lagrangian A-brane with a flat connection, as anticipated by the counting argument. On the other hand, if  $\mathcal{A}$  has rank two, a Gaussian integration can be performed in terms of an invertible submatrix of  $\mathcal{A}$ , called  $\mathcal{A}^{-1}$ . We are going to make use of the operation  $V^\perp \lrcorner (\cdot)$  defined [14] through

$$V^\perp \lrcorner (e^{\alpha_1} \dots e^{\alpha_k}) = \frac{1}{(3-k)!} \epsilon^{\alpha_1 \dots \alpha_3} e_{\alpha_{k+1}} \dots e_{\alpha_3}, k = 0 \dots 3.$$

This is essentially a Hodge star on the fiber, except it sends a  $k$ -form in the fiber into a  $3-k$ -vector (a section of  $\Lambda^{3-k}T$ ). The Hodge dual of the two-form  $\mathcal{A}_{\mu\nu}dy^\mu dy^\nu$  is a one-form on the fiber, to which the modified fiberwise Hodge star  $V^\perp \lrcorner$  associates a vector  $(V^\perp \lrcorner \mathcal{A})$ . In other words, the support of  $\mathcal{A}$  in the fiber is orthogonal to the vector  $(V^\perp \lrcorner \mathcal{A})$ , so that this vector entirely specifies the support

of the block-diagonal part of the curvature, enabling to perform the integration along the fiber, observing that the result is weighted by (the Poincaré dual of) a one-dimensional delta-function:

$$e^{F'} = \delta\left((\tilde{y}_\mu - \mathcal{S}_{\mu\nu}x^\nu)(V^\perp \lrcorner \mathcal{A})^\mu\right) \exp\left(\mathcal{A}_{\mu\nu}(dx^\mu \wedge dx^\nu) + \frac{1}{2}(\mathcal{S}dx)_\mu(\mathcal{A}^{-1})^{\mu\nu}(\mathcal{S}dx)_\nu\right).$$

Due to the presence of the first factor encoding the support of the Chern character, we will be able to treat  $F$  as a distribution in the sequel, so that the pull-back operation to the world-volume of the brane should be automatically built-in. We have therefore exhibited a five-dimensional object among the possible Fourier–Mukai transformations of holomorphic line bundles on the six-torus. The extension of this observation to more general Strominger–Yau–Zaslow fibrations and its mirror-symmetric interpretation will be the focus of the last section.

### 3 Action of Hamiltonian vector fields and non-Lagrangian A-branes

We start this section by briefly reviewing the definition of moment maps for the action of a group  $G$  on a symplectic manifold  $(M, \omega)$ . Let  $X$  be a generator of the corresponding Lie algebra; its action induces a vector field  $X^\sharp$ . If this vector field is Hamiltonian, then the corresponding Hamiltonian function  $\mu^X$  is called the moment map associated with the direction  $X$ . This is expressed as

$$d\mu^X = \iota_{X^\sharp}\omega.$$

The notion of moment map is a generalization of the Noether procedure in the context of symplectic geometry (for an ampler review see [25]).

Classical mechanics motivates the terminology, and provides the simplest and best known examples. One can for example compute the moment map for the action of  $\mathbf{R}^3$  by translations on  $\mathbf{R}^6$  endowed with the usual symplectic form  $\omega = dq^i \wedge dp_i$ . Consider the translation by  $\partial/\partial q^i$ . The associated vector field is  $\partial/\partial q^i$  and one reads off

$$\iota \frac{\partial}{\partial q^i} \omega = dp_i,$$

hence the moment map for translation by  $\partial/\partial q^i$  is the momentum  $p_i$ , thus recovering the result of the Noether procedure. The same exercise can be done for the action of  $\mathbf{R}^3$  by rotations, yielding the angular momentum as moment map.

The notion of moment map, after having emerged in global analysis, was extended by Atiyah and Bott [26] to gauge theory. In what follows, we are going to consider spaces of differential forms, in particular gauge connections, as infinite-dimensional symplectic manifolds (for a review of this framework in the context of topological branes, see [27]).

#### 3.1 Symplectic structures on gauge theory in the B-model

Bundles supported on an even-dimensional submanifold  $Y^{(2m)}$  of a Kähler manifold are naturally adapted to symplectic geometry on gauge theory [4, 5], in the sense that a natural pairing between one-forms follows from integrating along  $Y$ , filling the empty dimensions with the Kähler form:

$$\varpi(a, b) := \int_Y a \wedge b^* \wedge \omega^{m-1}.$$

The symplectic reduction of the space of connections is then the space of gauge orbits of the Hermitian Yang–Mills connections. This comes from the moment map associated to the gauge

transformations that preserve the holomorphicity conditions, namely shifts of the connection by  $\partial\phi + \bar{\partial}\phi$ , where  $\phi$  is a scalar function on  $Y$ . We can check that the moment map in the direction  $\phi$  is actually  $F \wedge \omega^{m-1}$ , and see that this is only a limiting case of a more general symplectic structure involving the connection. First let us see how the following quantity responds to a shift by  $h + h^*$  in the connection, where  $h$  is a  $(1, 0)$ -form:

$$\begin{aligned}\mu^\phi(A) &:= \int_Y \phi dA \wedge \omega^{m-1}. \\ d\mu^\phi(A) \cdot h &= \int_Y \phi \frac{d}{dt} \Big|_{t=0} (d(A + th) \wedge \omega^{m-1}) \\ &= \int_Y \phi dh \wedge \omega^{m-1} \\ &= - \int_Y (\partial + \bar{\partial})\phi \wedge h \omega^{m-1} \\ &= \iota_{\partial\phi + \bar{\partial}\phi} \varpi \cdot h.\end{aligned}$$

This means that  $\mu^\phi(A)$  is the component along the generator  $\phi$  of the moment map  $\mu(A)$  associated to the gauge transformations around the gauge configuration  $A$ . As a whole these components make for

$$\mu(A) = F(A) \wedge \omega^{m-1}.$$

The more general case comes from integrating the wedge-product of two one-forms against a gauge-field dependent kernel, which reduces to the previous one in the limit of small field strength.

$$\varpi'(a, b) := \int_Y a \wedge b^* \wedge \exp(i\omega + F).$$

In order for this to make sense as an Atiyah–Bott pairing, we need the form  $(\omega + iF)^{m-1}$  to be non-degenerate, which condition is implied by the deformed Hermitian Yang–Mills equation (1.1). Going through the same computation as above, we see that the moment map is actually the top-form from the power expansion of  $\exp(i\omega + F)$ :

$$\begin{aligned}\mu^{\phi'}(A) &:= \int_Y \phi \exp(i\omega + F(A)). \\ d\mu^{\phi'}(A) \cdot h &= \int_Y \phi \frac{d}{dt} \Big|_{t=0} (\exp(i\omega + (F + t dh))) \\ &= \int_Y \phi dh \wedge \exp(i\omega + F) \\ &= - \int_Y (\partial\phi + \bar{\partial}\phi) h \wedge \exp(i\omega + F) \\ &= \iota_{\partial\phi + \bar{\partial}\phi} \varpi' \cdot h\end{aligned}\tag{3.1}$$

These components combine into the following moment map, written as the top-form contribution from the integration kernel

$$\mu'(A) = \exp(i\omega + F(A)) \Big|_{\text{top}},\tag{3.2}$$

whose lowest non-trivial component in the small- $F$  limit is the moment map  $\mu(A)$  obtained from the zero-field strength limit of  $\varpi'$ . This fact corresponds to the deformed Hermitian Yang–Mills equations reducing to the ordinary ones in the same limit. The symplectic framework therefore parallels the results obtained from supersymmetry requirements in the B-model.

### 3.2 Moment map and non-Lagrangian A-branes

Symplectic geometry is involved in the A-model in a way that is more crucial for the geometry of the branes, but makes the symplectic structure on gauge theory less transparent (for odd-dimensional branes, the pairing between one-forms cannot be merely inherited from the ambient Kähler form). The list of symmetries of an A-brane includes the Abelian gauge group. It also contains the action of Hamiltonian vector fields, because the sequence (1.3) is preserved by this action, whether or not the brane is Lagrangian. The annihilation of  $\mathcal{L}Y$  by the Kähler form  $\omega$  is furthermore entangled with gauge symmetry as we are going to show.

Let  $h$  be a smooth real function on  $X$ . The vector field  $V_h := \omega^{-1}dh$  is Hamiltonian in the sense of the symplectic structure  $\omega$ , and preserves the kernel of  $\omega|_Y$ . As noticed by Kapustin and Orlov [8], it also acts on the connection by Lie derivation, so that allowed modifications to the connection are actually more involved than the mere shift by an exact form:

$$\delta A = \mathcal{L}_{V_h} A = \iota_{V_h} dA + d\iota_{V_h} A = \iota_{V_h} F + d(\iota_{V_h} A). \quad (3.3)$$

The first term vanishes in the case of special Lagrangian A-branes, because these are equipped with flat connections. In that case we are left with the second term, that falls into the class of Abelian gauge transformations. But on more general coisotropic branes, the field strength can actually be deformed under the action of vector fields. Defining a symplectic structure on the tangent space to connections on coisotropic branes, thus involving the field strength, can lead to richer deformations than in the special Lagrangian case. In particular, the moment map can be modified by powers of the field strength.

Let us review the derivation by Thomas [12] of the moment map in the special Lagrangian case. Let  $L$  be a special Lagrangian submanifold of  $X$ . There, one considers the variation<sup>1</sup> of the integral of the holomorphic form against a scalar function  $h$ , under the action of a Hamiltonian vector field:

$$\begin{aligned} V_u &:= \omega^{-1}du. \\ V_u \int_L h \Omega &= \int_L h \frac{d}{dt} \Big|_{t=0} (\Phi_t^* \Omega) = \int_L h d(\iota_{V_u} \Omega) = \int_L dh \wedge (\iota_{V_u} \Omega) \\ &=: \varrho(dh, du), \end{aligned} \quad (3.4)$$

where the pairing  $\varrho$  is defined by its action on one-forms as

$$\varrho(a, b) = \int_L a \wedge (\iota_{\omega^{-1}b} \Omega).$$

When written in terms of the phase  $e^{i\theta}$  of the special Lagrangian condition (1.4), the pairing  $\varrho$  may be identified as a metric on the tangent space to the space of flat connections, since it is symmetric in its arguments:

$$\varrho(a, b) = \int_L \cos \theta (a \wedge *b),$$

so that the above functional is the component of the moment map along the generator  $h$ . Altogether we obtain for special Lagrangian A-branes:

$$\mu = \text{Im } \Omega|_Y.$$

Going beyond the Lagrangian case, consider for definiteness a five-dimensional submanifold  $Y$  of the Calabi–Yau manifold  $X$ . The natural integral on  $Y$ , in which the genuinely coisotropic

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<sup>1</sup>the action of the Hamiltonian vector field  $V_u$  is the flow  $\Phi_t$  it generates.

deformation of the connection read from (3.3) can modify the action (3.4), is the following:

$$\int_Y h \left( d(\iota_{V_u} \Omega) \wedge F + \Omega \wedge d(\delta A) \right) = \int_Y h \left( d(\iota_{V_u} \Omega) \wedge F + \Omega \wedge d(\iota_{V_u} F) \right).$$

We see that the piece from ordinary gauge transformations is annihilated by the derivation, and we are left with the new deformation of the connection. Integrating by parts we obtain our candidate for the pairing on the tangent space to the space of connections on the A-brane:

$$\begin{aligned} \int_Y h \left( d(\iota_{V_u} \Omega) \wedge F + \Omega \wedge d(\iota_{V_u} F) \right) &= \int_Y h d\iota_{V_u} (\Omega \wedge F) \\ &= \int_Y dh \wedge (\iota_{V_u} (\Omega \wedge F)). \end{aligned}$$

The natural pairing to define at this point indeed reads

$$\varrho'(a, b) = \int_Y a \wedge (\iota_{\omega^{-1}b} (\Omega \wedge F)).$$

The proportionality factor between the five-form  $\Omega \wedge F$  and the volume form on  $Y$  is then the suitable slope for coisotropic branes. Its very existence moreover ensures that  $\varrho'$  actually defines a symplectic structure on the tangent space:

$$(\Omega \wedge F)|_Y =: e^{i\theta} \text{vol}_Y.$$

Taking into account the deformation of non-flat connections thus extends the picture from special Lagrangian A-branes to coisotropic branes and induces the following moment map:

$$\mu'(A) = \text{Im} (\Omega|_Y \wedge F). \quad (3.5)$$

This quantity should be mirror to the moment map studied in the B-model, provided stable objects are mapped to stable objects by mirror symmetry. The symmetries of coisotropic A-branes thus fit into symplectic geometry on gauge connections to generate a candidate for a mirror of stable holomorphic line bundles, that is not tied by the special Lagrangian assumption. This modification of the moment map by gauge fields in the A-model provides a differential-geometric explanation of the world-sheet result of [7]. Let us now turn to an explicit verification by T-duality of the exchange between (3.2) and (3.5) under mirror symmetry.

## 4 T-duality and exchange of modified pure spinors

The above completion of the symplectic picture by non-Lagrangian A-branes has taught us that for any stable brane supported on a submanifold  $Y$  of a Calabi–Yau manifold  $X$ , mirror symmetry exchanges differential forms of maximal degree, where the possible contribution of the gauge fields comes from the Chern character, namely (1.2) in the A-model and (1.1) in the B-model. Without the contribution from the Chern characters, these two quantities reduce to two pure spinors that have been shown in [14] to be exchanged by mirror symmetry, for any manifolds of  $SU(3)$ -structure possessing a  $T^3$ -fibration. Combining these two objects together, we arrive at a proposal for the mirror exchange between two pure spinors modified by a connection supported on D-brane world-volume.

We consider generic six-dimensional manifolds  $X$  that admit  $T^3$ -fibrations: strictly speaking there is no reason to expect that backgrounds with fluxes should admit these (and in some known cases they simply do not), however this assumption is important computationally. Note that this assumption could be relaxed when giving the prescription for the mirror symmetry with  $H$ -flux but



without branes.

As just mentioned, T-duality along the  $T^3$ -fiber amounts to the exchange of the two pure spinors  $e^{i\omega}$  and  $\Omega$ . Since the exchange still holds in the presence of a  $B$ -field, incorporating branes into the picture is not too hard. Indeed, by properly defining the gauge field  $F$  in such a way that it extends over the whole six-manifold  $X$  irrespectively of the dimensionality of the brane (working with the elements of  $Q \in K(X)$ ), we work with the gauge-invariant combination  $B - F$  formally treating it as a  $B$ -field, while bearing in mind the T-duality transformation of the gauge fields.

#### 4.1 T-duality for closed strings

First, we quickly review the T-duality for the case without gauge bundles, following [14] (with a slight change of conventions in the names of the coordinates, for which we rather follow [10]). The six-dimensional manifold will be taken to be a  $T^3$ -fibration over a base  $B_3$ . Coordinates on the base will be denoted by  $(x^1, x^2, x^3)$ , and those on the fiber by  $(y^1, y^2, y^3)$ . All the quantities will only depend on the  $x$  coordinates. Then  $i, j, k, \dots$  are used in the three-dimensional  $x$  subspace and  $\alpha, \beta, \gamma, \dots$  are used in the three-dimensional  $y$  subspace;  $a, b, c, \dots$  and  $a', b', c', \dots$  are used in the three-dimensional real  $x$  and  $y$  frame spaces. As for the six-dimensional covariant notation,  $\mu, \nu, \dots$  are used in the total six-dimensional space for real coordinates ( $dx^\mu = (dx^i, dy^\alpha)$ ), while  $m, n, \dots$  are reserved for holomorphic/antiholomorphic indices; finally,  $A, B, C, \dots$  are indices in the total three-complex-dimensional frame space.

The metric and  $B$ -field are given by

$$ds^2 = g_{ij} dx^i dx^j + h_{\alpha\beta} e^\alpha e^\beta = G_{\mu\nu} dx^\mu dx^\nu$$

$$B = \frac{1}{2} B_{ij} dx^i \wedge dx^j + B_\alpha \wedge (dy^\alpha + \frac{1}{2} \lambda^\alpha) + \frac{1}{2} B_{\alpha\beta} e^\alpha \wedge e^\beta$$

where  $\lambda^\alpha = \lambda_i^\alpha dx^i$ ,  $B_\alpha = B_{i\alpha} dx^i$ , and we have defined

$$e^\alpha \equiv dy^\alpha + \lambda^\alpha .$$

Before passing to pure spinors we need to introduce the two basic objects, namely a two-form  $\omega$  and a three-form  $\Omega$  satisfying  $\omega \wedge \Omega = 0$  and  $i\Omega \wedge \bar{\Omega} = (2\omega)^3/3!$ . To do this, we start from  $(1, 0)$  vielbein, which in turn defines an almost complex structure

$$E^A = i e_i^a dx^i + V_\alpha^{a'} e^\alpha \quad (4.1)$$

where  $A = a = a'$  goes from 1 to 3; the corresponding  $(0, 1)$  vielbein is  $E^{\bar{B}} = \overline{E^B}$ . The holomorphic three-form reads  $\Omega = E^1 \wedge E^2 \wedge E^3 = \frac{1}{6} \epsilon_{ABC} E^A \wedge E^B \wedge E^C$  and the fundamental two-form  $\omega$  is

$$\omega = \frac{i}{2} \delta_{AB} E^A \wedge E^{\bar{B}} = -V_{i\alpha} dx^i \wedge e^\alpha .$$

T-duality leaves  $g_{ij}$  and  $B_{ij}$  invariant, while the components of the metric and  $B$ -field with legs along the dualized directions transform as

$$h_{\alpha\beta} \longleftrightarrow \hat{h}^{\alpha\beta} ; \quad B_{\alpha\beta} \longleftrightarrow \hat{B}^{\alpha\beta} ; \quad B_\alpha \longleftrightarrow \lambda^\alpha . \quad (4.2)$$

We can now introduce the vielbein  $\hat{V}^{a\alpha}$  of the T-dual metric  $\hat{h}^{\alpha\beta}$ , that satisfies  $\hat{V}^{a\alpha} \hat{V}^{a\beta} = \hat{h}^{\alpha\beta}$ :

$$\hat{V}^{a\alpha} = \left( \frac{1}{h + B} \right)^{\alpha\beta} V_\beta^a = V_\beta^a \left( \frac{1}{h - B} \right)^{\beta\alpha}$$

where  $V_\beta^a$  is naturally the original vielbein. Writing down the inverse

$$\hat{V}_\alpha^a \equiv \hat{h}^{\alpha\beta} \hat{V}^{a\beta} = (h - B)_{\alpha\beta} V^{a\beta} = V^{a\beta} (h + B)_{\beta\alpha} .$$

We can complete the T-duality rules by giving the transformations of the vielbeine:

$$V_\alpha^a \longleftrightarrow \hat{V}^{a\alpha} ; \quad V^{a\alpha} \longleftrightarrow \hat{V}_\alpha^a . \quad (4.3)$$

We will mostly work in the case where the  $B$ -field is purely of base-fiber type in frame indices. Transformation (4.2) shows that this condition is conserved by T-duality, and  $\hat{h}^{\alpha\beta} = h^{\alpha\beta}$ . Consequently,  $\hat{V}^{a\alpha} = V^{a\alpha}$  and  $\hat{V}_\alpha^a = V_\alpha^a$ . T-duality then only amounts to moving fiber indices up and down (still exchanging  $B_\alpha$  and  $\lambda^\alpha$  though).

First we do the easier case, in which there is neither  $B$ -field nor  $\lambda$  twisting of the  $T^3$  bundle. The basic idea is that  $\Omega$  can be written in a sense as an exponential of the almost complex structure  $\omega_\mu{}^\nu$  applied to a degenerate three-form  $\epsilon_{ijk} dx^i dx^j dx^k$ , that can be thought of as the holomorphic three-form in the large complex-structure limit. More explicitly, we expand

$$\Omega = E^1 \wedge E^2 \wedge E^3$$

using the expression for the holomorphic vielbein in (4.1). We obtain four terms, with  $dx^3$ ,  $dx^2 e$ , and so on. As in the example of the torus, we use the duality-friendly operation  $V^\perp \lrcorner (\cdot)$ . Using the fact that the lower  $e_\alpha$  are indeed vectors  $\partial_\alpha \equiv \partial / \partial q^\alpha$ , we map  $\Omega$  into a sum of  $(k, k)$  tensors - objects with  $k$  indices up and  $k$  down, the latter being along the fiber. The sum can be expressed as an exponent of  $V_i^\alpha e_\alpha dx^i$ , which is the complex structure. According to (4.3) the action of T-duality now simply raises and lowers the  $\alpha$  index: the tangent bundle (in the fiber direction) of the initial manifold is equal to the cotangent bundle (again in the fiber direction) of the T-dual manifold. As a result, the complex structure gets now mapped to  $V_{i\alpha} e^\alpha dx^i$ , the fundamental two-form  $\omega$ , so that

$$T(V^\perp \lrcorner \Omega) = \frac{i}{3!} e^{i\omega} .$$

To incorporate the case with non-zero  $B$ -field and corresponding twisting of the  $T^3$ -fiber with a connections  $\lambda$ , it is convenient to recall that  $e^{i\omega}$  and  $\Omega$  are Clifford(6,6) spinors, and that the forms act on these as combinations of gamma matrices. In particular, due to purity of  $\Omega$ , we have the identity  $\gamma^m \Omega = \gamma_m \Omega = 0$ . Taking  $B$  for the time being to be only of base-fiber type, we note that due to  $\gamma^\alpha \Omega = i \gamma^i V_i^\alpha \Omega$  we have  $e^B \Omega = e^{iB_\alpha \wedge V^\alpha} \Omega$ . T-duality then gives

$$\frac{i}{3!} T(e^{i\omega}) = V^\perp \lrcorner (e^B \Omega) e^{-B_\alpha \lambda^\alpha} ,$$

$$T(\Omega) = \frac{i}{3!} V^\perp \lrcorner (e^B e^{i\omega}) e^{B_\alpha \lambda^\alpha} .$$

This can be presented in a better form as

$$T : \quad \frac{i}{3!} e^{i\omega+B} \longrightarrow \int_{T^3} e^{\mathcal{P}} e^B \Omega, \quad (4.4)$$

and the same with  $e^{i\omega}$  and  $\Omega$  exchanged. We denoted by  $\mathcal{P} = e^\alpha \wedge \hat{e}_\alpha$  the connection on the “twisted” Poincaré bundle. Due to it being inert under T-duality, adding  $B_{ij}$  is trivial, so  $B$  in (4.4) has either both legs along the base, or base-fiber components; the case of a  $B$ -field with all legs along  $T^3$  leads upon T-duality to non-geometrical situations and will not be considered here.

## 4.2 Open-string T-duality

Having reviewed the closed-string case, we are ready to turn to the branes. Before we do so, we note that for the above construction as well as for the applications, a crucial use was made of similarity between pure spinors and Ramond–Ramond forms, both transforming as spinors of  $\text{Clifford}(d, d)$ . In [16], the brane charges have also been treated as elements of  $\text{Clifford}(d, d)$ , and our treatment of T-duality will be following this treatment closely. We will use the coupling of Ramond–Ramond fields to D-branes to find the suitable modification of pure spinors for the open-string case. For the time being, we ignore the gravitational corrections and take  $Q = e^F$ ; we mostly concentrate on the case of Abelian branes.

As mentioned, we treat the gauge curvature  $F$  of all branes as a six-dimensional object. One way of doing so is to put together the gauge field on the brane and the transverse scalars into a six-vector. When restricting to the brane, components of  $F$  with two longitudinal indices will naturally be the gauge curvature on the brane, while components of  $F$  with mixed indices stand for the derivatives of the transverse scalars (covariantized both by the connection on the normal bundle). Components of  $F$  with two transverse indices would finally be made of transverse scalars only, and vanish for a single brane with an Abelian connection. In addition to this, as for the  $T^3$ -fibered metric, we take the fields to depend only on (the subset of) the base coordinates  $x$ . The branes generically wrap both the base and the fiber directions.

Turning on  $F$ , the first thing we notice is that it passes through  $V^\perp \lrcorner (\cdot)$  just like the  $B$ -field, whenever it is of base-fiber type. Indeed, adding  $F_{ij}$  components is again trivial, while as can be seen from earlier discussion,  $F_{\alpha\beta} = 0$  in the case of stable Abelian branes, and so we will not need to consider these. Thus, suppressing the  $i$  index and ignoring momentarily the  $F_{ij}$  component, which are inert under T-duality, we can write with a slight abuse of notation the gauge curvature as  $F = (F_\alpha, f^\alpha)$  with a natural constraint  $F_\alpha f^\alpha = 0$ . Once more, we observe that the  $i$ – $\alpha$  split has nothing to do with the D-brane longitudinal–transverse split. In a static-gauge picture,  $F_\alpha$  here would denote the part of the field strength with one leg along the base of the  $T^3$ -fibration and with the other along the fiber;  $f^\alpha$  stands for derivative of transverse scalars parameterizing the directions along the  $T^3$  fiber.

Under T-duality along all of  $T^3$  we simply have  $F_\alpha \longleftrightarrow f^\alpha$ , and  $F_\alpha V^\alpha + V_\alpha f^\alpha$  is invariant. Notice that T-duality along  $T^3$  sends the connections with even-dimensional support to those with odd-dimensional ones vice versa (as it should). The outcome of all this is that we can simply dress both pure spinors in (4.4) with  $Q = e^F$ , and have the same exchange of pure spinors as under the closed-string mirror symmetry:

$$T : \quad \frac{i}{3!} (Q \wedge e^{i\omega}) e^B \longrightarrow \int_{T^3} e^{\mathcal{P}} e^B (Q \wedge \Omega). \quad (4.5)$$

One may in fact put the field strength  $F$  with its dual  $\hat{F}$  into a single  $\text{Clifford}(6, 6)$  object [16]. We did not need to do this in order to obtain the explicit transformation above but this is very much the underlying logic and is important for considering the general (non-Abelian) case. Note that these considerations naturally lead to the definition of a generalized complex submanifold as given by a product of the tangent bundle to the submanifold with the *co*-normal bundle [2] (see also [28, 29]). The formula (4.5) is consistent with the conjectures on mirror symmetry on Calabi–Yau manifolds with bundles [17], and may also be regarded as a general statement on closed-open string duality. Turning on and off  $F$  and  $H = dB$  respectively, one may relate different geometries with only the  $H$ -flux or with a single D-brane.

We have been ignoring the gravitational parts of  $Q$ , but may recall now that  $Q = \sqrt{\hat{A}(X)} e^{iF}$ , so that it is tempting to speculate that the complete  $Q$  should be the factor modifying the pure spinor;

moreover one can check that (4.5) is compatible with the general T-duality transformations on the D-branes charges as elements of  $K$ -theory. The exchange of odd-even dimensional world-volumes is well-understood on  $K$ -theory level. Here we emphasize once more that the discussion is at the level of elements of Clifford(6,6) and that all the objects are defined in the bulk. In fact we note that the modified pure spinor  $\varphi \wedge Q$  formally looks just like the bulk coupling of Ramond–Ramond forms to brane charges, and in this sense the mirror/T-duality exchange (4.5) is very natural.

As for the stability conditions, we are not pursuing here the question of possible generalizations of the calibrations and searching for new stable branes in the flux backgrounds. So taking  $H = 0$  (while still having  $B \neq 0$  extension of the Fourier–Mukai derivations, which are typically performed with vanishing  $B$ -field; however see [30] for a quantum mechanical model of D-brane and mirror symmetry in the presence of a  $B$ -field), one can check that the modified pure spinors  $\varphi \wedge Q$  indeed do give the moment maps (3.5) and (3.2), and the stability equations for both A and B-type branes.

We conclude by a conjecture about multiple D-brane wrappings, and thus the inclusion of non-Abelian connections. These may have  $F^{\alpha\beta} \neq 0$  (going to the static gauge again, on the brane  $F^{\alpha\beta} = [\phi^\alpha, \phi^\beta]$ , where  $\phi^\alpha$  are the scalars along  $T^3$ ). As argued in [16, 31] for couplings to Ramond–Ramond fields  $C$  (at least for the case with  $B = 0$ ), consistency with T-duality would require then replacing the wedge product  $C \wedge Q$  by a Clifford multiplication. Given that for pure spinors as well we are dealing with elements of Clifford(6,6) this is a natural expectation. At the level of world-volume (i.e. stability equations) this will replace the deformed Hermitian Yang–Mills equations by their counterpart with Hitchin terms containing commutators of scalars. On the B side, such equations have indeed been studied (see [32]). On the A side, there will be no modifications containing the scalars for the coisotropic (including special Lagrangian) branes. However there would seem to appear a new one-dimensional equation involving  $\Omega_{i\alpha\beta} F^{\alpha\beta}$ , which could be non-trivial for manifolds with  $b_1(M) \neq 0$ , and would indeed restore the democracy between all the odd- and even-dimensional cycles on A and B sides respectively.

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